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LETTER TO THE EDITOR

SO(2, 1)-invariant systems and the Berry phase

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Abstract. The general quadratic time-dependent quantum Hamiltonian is analysed using a time-dependent realisation of its SO(2, 1) invariance Lie algebra. Several interesting features of this procedure are pointed out. As an example, we easily calculate dynamical and geometrical (Berry) phases using only an algebraic procedure. An incorrect result previously published by another author is also pointed out.

We present in this letter an exact calculation of dynamical and geometrical phases [1] of a general quadratic time-dependent quantum Hamiltonian. Here, exact calculation means that we do not confine ourselves to the adiabatic case and we calculate the geometrical phase for arbitrary changes in time. Indeed, when restricting the expression to the adiabatic limit we shall recover Berry's result. This general example can be added to those presented by Berry and Hannay [2] in their general discussion of non-adiabatic angles. Of course, our result proves once again that the sum of the dynamical and geometrical contributions to the total phase is preserved when a quantum system undergoes a change in parameter space along a closed path in time. We should also point out that there exists a relationship between our group-theoretical based calculations and the squeezed-state formalism as we shall discuss in a forthcoming report [3].

Let us start with the most general quadratic Hamiltonian written in canonical form:

$$H(t)=\beta_3(t)\frac{p^2}{2m}+\frac{\omega_0}{2}\beta_2(t)(xp+px)+\beta_1(t)\frac{m\omega_0^2}{2}x^2 \tag{1}$$

where the β are real functions of time and for $t=t_0$: $\beta_1(t_0)=\beta_3(t_0)=1$ and $\beta_2(t_0)=0$. Consider the usual creation and annihilation operators:

$$a=(2m\omega_0\hbar)^{-1/2}[m\omega_0 x+\mathrm{i}p] \tag{2a}$$

$$a^+=(2m\omega_0\hbar)^{-1/2}[m\omega_0 x-\mathrm{i}p]. \tag{2b}$$

In terms of these operators we can write (1) as:

$$\begin{aligned}H(t)&=\hbar\omega_0\left[(\beta_1+\beta_3)K_0+\left(\frac{\beta_1-\beta_3}{2}+\mathrm{i}\beta_2\right)K_++\left(\frac{\beta_1-\beta_3}{2}-\mathrm{i}\beta_2\right)K\right]\\&=\hbar\omega_0(f_0(t)K_0+f(t)K_++f^*(t)K_-)\end{aligned} \tag{3}$$

and the K-operators are given by:

$$K_0=\tfrac{1}{2}(a^+a+\tfrac{1}{2}) \tag{4a}$$

$$K_+=\tfrac{1}{2}a^+a^+ \tag{4b}$$

$$K_-=\tfrac{1}{2}aa. \tag{4c}$$

They close the SO(2, 1) Lie algebra:

$$[K_0, K_\pm] = \pm K_\pm \tag{5a}$$

$$[K_+, K_-] = -2K_0. \tag{5b}$$

Now we look for a set of *time-dependent* creation and annihilation operators. The obvious link with the SO(2, 1) coherent-state representation of such a set of operators will be discussed elsewhere [3]. For our present purposes it is enough to know that

$$a(t) = \mu(t)a + \nu(t)a^+ \tag{6a}$$

$$a^+(t) = \nu^*(t)a + \mu^*(t)a^+ \tag{6b}$$

and, indeed, $|\mu|^2 - |\nu|^2 = 1$ in order to keep the usual canonical commutation relation unchanged: $[a(t), a^+(t)] = 1$. Also $\mu(t_0) = 1$; $\nu(t_0) = 0$.

A SO(2, 1) representation similar to (4*a*)-(4*c*) can be constructed with the new *time-dependent* operators (6*a*) and (6*b*) as

$$M_0(t) = \tfrac{1}{2}(a^+(t)a(t) + \tfrac{1}{2}) \tag{7a}$$

$$M_+(t) = \tfrac{1}{2}a^+(t)a^+(t) \tag{7b}$$

$$M_-(t) = \tfrac{1}{2}a(t)a(t) \tag{7c}$$

which indeed fulfil the SO(2, 1) Lie algebra (5*a*) and (5*b*). The Heisenberg invariance condition $\mathscr{D}(0) = 0$, where

$$\mathscr{D}(\ ) = \frac{\partial}{\partial t} - \frac{\mathrm{i}}{\hbar}[\ , H(t)] \tag{8}$$

is now imposed on $M_0(t)$ (notice that $M_0(t_0) = K_0 = (2h\omega_0)^{-1}H(t_0)$). Using the Lewis-Riesenfeld theorem [4] one can show that the eigenvalues of $M_0(t)$ are time independent. Applying (8) to $M_0(t)$ and after some tedious calculations, we obtain the general form of $\mu(t)$ and $\nu(t)$:

$$\mu(t) = \frac{1}{2(m\omega_0\beta_3)^{1/2}}\left(\frac{1}{\sigma} + m\omega_0\beta_3\sigma - \mathrm{i}m\sigma\Lambda\right) \tag{9a}$$

$$\nu(t) = \frac{1}{2(m\omega_0\beta_3)^{1/2}}\left(\frac{1}{\sigma} - m\omega_0\beta_1\sigma - \mathrm{i}m\sigma\Lambda\right) \tag{9b}$$

where

$$\Lambda = \Lambda(t) = \frac{\dot\sigma}{\sigma} + \frac{\dot\beta_3}{2\beta_3} - \omega_0\beta_2.$$

The auxiliary function $\sigma(t)$ is a solution of the differential equation

$$\ddot\sigma(t) + \Omega^2(t)\sigma(t) = \frac{1}{m^2\sigma^3(t)} \tag{10}$$

where

$$\Omega^2(t) = \omega_0^2(\beta_1\beta_3 - \beta_2^2) + \omega_0\frac{\beta_2\dot\beta_3 - \beta_3\dot\beta_2}{\beta_3} + \frac{\ddot\beta_3}{2\beta_3} - \frac{3}{4}\frac{\dot\beta_3^2}{\beta_3^2} \tag{11}$$

which is, of course, the classical frequency of the classical equations of motion.

Now we re-express $H(t)$ in terms of the time-dependent operator basis $M_0(t)$ and $M_\pm(t)$. We can do that by using (4), (6), (9) and $M_\pm(t)=M_1(t)\pm \mathrm{i}M_2(t)$. We finally obtain:

$$H(t)=2\hbar(\vartheta_0(t)M_0(t)+\vartheta_1(t)M_1(t)+\vartheta_2(t)M_2(t)) \tag{12}$$

where

$$\vartheta_0(t)=\tfrac{1}{2}m\sigma^2\left(\vartheta_2^2(t)+\omega_0^2(\beta_1\beta_3-\beta_2^2)+\frac{1}{m^2\sigma^4}\right) \tag{13a}$$

$$\vartheta_1(t)=\tfrac{1}{2}m\sigma^2\left(\vartheta_2^2(t)+\omega_0(\beta_1\beta_3-\beta_2^2)-\frac{1}{m^2\sigma^4}\right) \tag{13b}$$

$$\vartheta_2(t)=-\left(\frac{\dot\sigma}{\sigma}+\frac{\dot\beta_3}{2\beta_3}\right)=-\frac{\mathrm{d}}{\mathrm{d}t}\log(\sigma\beta_3^{1/2}). \tag{13c}$$

It is also trivial to show the following SO(2, 1) relationship between the frequencies defined in (13*a*)-(13*c*):

$$\vartheta_0^2-\vartheta_1^2-\vartheta_2^2=\omega_0^2(\beta_1\beta_3-\beta_2^2) \tag{14}$$

which is the geometrical measure of the hyperbolic SO(2, 1)-parameter manifold. In order to see the beautiful relationship with the angles, let us calculate in a different framework the following dynamical and geometrical quantities:

$$\Omega_d=-\frac{1}{\hbar}\langle n,t|H(t)|n,t\rangle \tag{15a}$$

$$\Omega_g=\mathrm{i}\langle n,t|\partial/\partial t|n,t\rangle \tag{15b}$$

where $|n,t\rangle$ is a time-dependent eigenstate of $M_0(t)$ with time-independent eigenvalue $\frac{1}{2}(n+\frac{1}{2})$. Thus, we easily find:

$$\Omega_d=-\vartheta_0(t)(n+\tfrac{1}{2}) \tag{16a}$$

$$\Omega_g=\vartheta_1(t)(n+\tfrac{1}{2}). \tag{16b}$$

Several consequences can be drawn from these results.

(i) The SO(2, 1) formalism allows us to write the Hamiltonian in the time-dependent representation as a linear combination of M operators whose coefficients are the relevant frequencies which give rise to the dynamical and geometrical phases upon integration over time. These frequencies are exact and no adiabatic hypothesis has been made.

(ii) It is easy to show that $\Omega_d+\Omega_g=-(n+\frac{1}{2})/m\sigma^2$ (the Lewis frequency). This sum is preserved as an invariant of the physical system as has been demonstrated in [2].

(iii) Although the general Hamiltonian (1) can be transformed at the classical level into a simple time-dependent frequency harmonic oscillator through a canonical transformation, the quantum theory is more subtle. In particular, due to the time dependence of the theory, there is no trivial way to relate the states in both representations. Therefore, one cannot assign physical properties of one system to the other without performing the explicit calculations. In a recent paper, Morales [5] claims to give the exact correct expression for the Berry phase using only the assumption of canonically related physical systems and fails to give the correct result due to the above-mentioned reasons. This result is in fact given by our expressions (13*b*), (15*b*) and (16*b*).

The method used in this letter can be applied to other physical systems exhibiting SO(2, 1) invariance, such as the conformal oscillator. These and other features related to coherence will be discussed in a forthcoming paper.

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